Cosmological Correlators in Momentum Space

Chandramouli Chowdhury

STAG Research Centre
University of Southampton

Motives in QFT and String: IPhT-Saclay

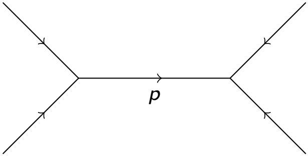
Collaborators & Projects:-

- 1904.10043: Savan Kharel & Soner Albayrak
- 2001.06777: Savan Kharel & Soner Albayrak
- 2305.18529: Kajal Singh
- 2312.yyyy: Arthur Lipstein, Jiajie Mei, Ivo Sachs, Pierre Vanhove
- 24xx.yyyy: Arthur Lipstein, Ricardo Monteiro, Silvia Nagy, Kajal Singh
- 24xx.yyyy: Pratyusha Chowdhury, Kajal Singh

Motivation for Scattering Amplitudes

1. Scattering amplitudes are some of the most basic questions one can ask about quantum field theory and gravity.
2. These are directly related to physical observables that are calculable by experiments.
3. Typical approach is via Feynman Diagrams.
4. They are diagrammatic rules for describing terms in perturbation theory.

1. These are typically evaluated for (time-ordered) correlation functions + LSZ.
2. For convenience they are usually evaluated in momentum space.
3. A very simple non-trivial Feynman diagram corresponding to a 2→2 scattering is given by

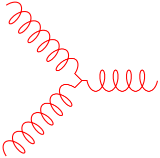


This diagram is given as $\frac{g^2}{p^2+m^2}$ (involve off-shell particles) .

4. This is not a physical observable, but it is the central building block of one.

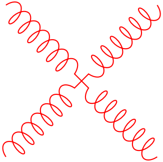
1. The example shown in the previous slide was for a simple scalar theory.
2. For experimental purposes: More complicated theories (or are they?)!
3. One such example is Yang-Mills theory (1954) which is described by a gauge symmetry.
4. Another realistic theory – Gravity (1600's)!
5. Are the rules as simple for them?

1. Vertices for YM theory



$$= f^{abc}\left[(p_1 - p_2)_\lambda \eta_{\mu\nu} + (p_2 - p_3)_\mu \eta_{\nu\lambda} + (p_3 - p_1)_\nu \eta_{\lambda\mu}\right]$$

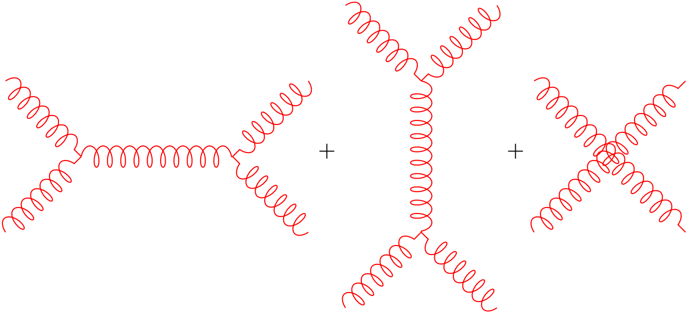
2. Similarly also a 4-pt vertex



$$= ff\eta + \cdots$$

3. Vertices for Gravity are extremely complicated.

1. $2 \to 2$ scattering in YM gives



$$+ \quad + \qquad (1)$$

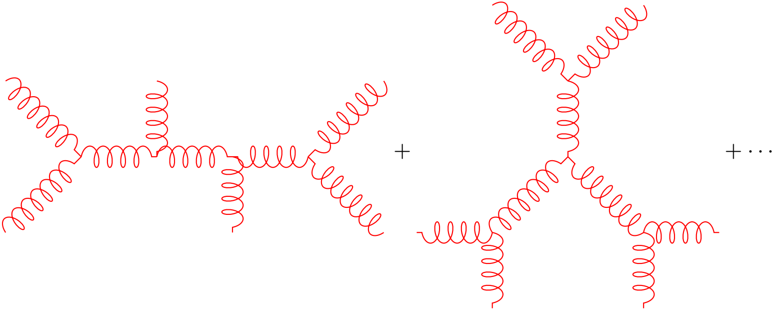
2. Not too difficult, example in a QFT course.

$$\frac{1}{s}(\cdots) + \frac{1}{t}(\cdots) + \text{cross}$$

1. Soon number of diagrams increase and get more complicated.

$$g + g \to g + g + g \qquad :25 \text{ diagrams}$$
$$g + g + g \to g + g + g \qquad :220 \text{ diagrams}$$

2. 6-gluon scattering:



1. For example, a $2 \to 3$ point amplitude gives this from a brute force computation

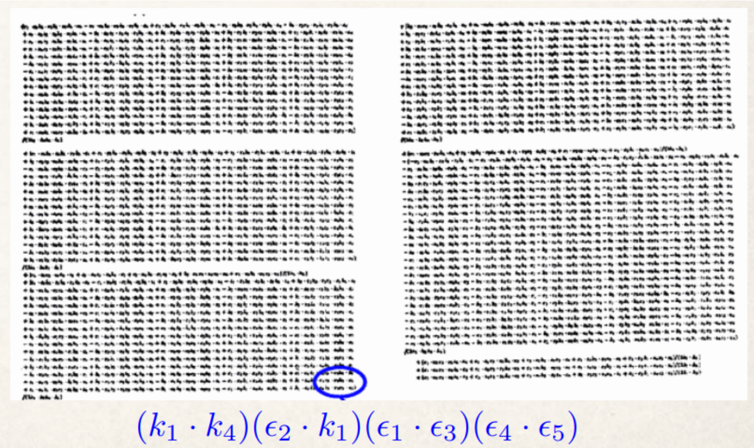


Figure: $gg \to ggg$ Scattering
Credit for image: Jaroslav Trnka

Parke-Taylor Formula

1. The $2 \to 4$ point amplitude was evaluated by Parke and Taylor in 1985 and constituted of 220 Feynman Diagrams $\sim$ 100 pages of computation!
2. Summarized the answer in a short paper (the whole paper is the answer)!

Figure: 6 point Gluon Amplitude.
Credit for image: Jaroslav Trnka

1. They even mentioned the following line in their 1985 paper "*we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight*".

2. And within a year (1986) they guessed a simple answer!

$$\mathcal{M}_6 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} \tag{2}$$

These $\langle 12\rangle$ are simple functions of momenta p_1, p_2 and are known as Spinor-Helicity variables [van-der-Waerden (1929)].

3. Checked by them for $2 \to 2$, $2 \to 3$, $2 \to 4$, natural conjecture for n

$$\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

1. There exists a similar problem for Gravity, where the 4-point function has 2580 terms [DeWitt, 1967]

2. However many cancellations occur and the final expressions are again simple, *"... a large amount of cancellation between terms still has to be dug out of the algebra, and this, combined with the fact that the final results are ridiculously simple, leads one to believe that there must be an easier way."*

Easier Way?

1. This "easier way" was developed in 2003 by Britto-Cachazo-Feng-Witten.
2. Uses On-Shell methods instead of Off-Shell diagrams.
3. Eventually led to a lot more developments and the recent "Amplitudes program".

Going away from flat space

1. Our universe is not flat! We live in an accelerating universe.
2. Computations of correlation functions are typically much harder in FLRW/dS.
3. This requires an understanding of the wave function itself (also contains the flat space scattering amplitude!).
4. Wave functions are defined via [Hartle-Hawking (1983)]

$$\Psi[\varphi] = \int_{\phi(-\infty)=0}^{\phi(0)=\varphi} D\phi \; e^{iS[\phi]} \, . \tag{3}$$

   These are also equivalent to computing correlators in AdS [Maldacena (2002)].
5. We'll discuss this in momentum space and some tools from flat space amplitudes.

Wave Function

$$\Psi[\varphi] = \int_{\phi(-\infty)=0}^{\phi(0)=\varphi} D\phi \; e^{iS[\phi]}$$

1. ϕ is divided into two parts:

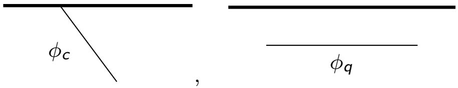
ϕ_c: satisfying free EOM & boundary conditions
ϕ_q: zero boundary conditions & all interactions

2. Expand action as $S = S_{quad}[\phi_c] + S_{quad}[\phi_q] + S_{int}[\phi_c, \phi_q]$. For example in $\lambda\phi^4$,

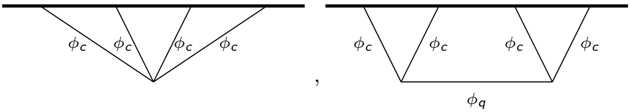
$$\Psi \sim \int D\phi_q e^{iS_{quad}[\phi_q]} \left[1 + \lambda\phi_c\phi_c\phi_c\phi_c + O(\lambda^2)\right]$$

Wave Function

1. Diagrammatic rules for the path integral



2. Computation of the path integral can be organized in terms of Witten diagrams.



3. In any diagram, only 3-momenta is conserved.

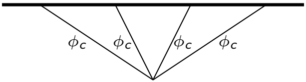
Simple Examples

1. For conformally coupled scalars we can map (A)dS $\to$ Flat space with a boundary at $(z)t = 0$. Generally interactions pick up a $(z)t$ dependence.

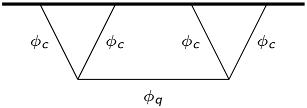
2. Propagators are

$$\phi_c = e^{ikt},$$
$$G(t,t') = \frac{1}{2k}\left[\theta(t-t')e^{ik(t-t')} + \theta(t'-t)e^{ik(t'-t)} - e^{ik(t+t')}\right] \tag{4}$$

3. Dirichlet boundary conditions: $G(0,t') = G(t,0) = 0$



$$= \int_0^\infty dt e^{i(k_1+k_2+k_3+k_4)} = \frac{1}{k_1+k_2+k_3+k_4}$$



$$= \int_0^\infty dt_1 dt_2 e^{ik_{12}t_1} e^{ik_{34}t_2} G(t_1, t_2, \vec{k}_1 + \vec{k}_2)$$

$$= \frac{1}{2|\vec{k}_1 + \vec{k}_2|} \int_0^\infty dt_1 dt_2 e^{ik_{12}t_1} e^{ik_{34}t_2} \Big[\Theta(t_1 - t_2) e^{ik(t_1-t_2)} + \Theta(t_2 - t_1) e^{ik(t_2-t_1)} - e^{ik(t_1+t_2)}\Big]$$

- Above equation : sum of 3-terms. However, final answer is simple!

$$= \frac{1}{(k_{12} + k_{34})(k_{12} + k)(k_{34} + k)}$$

where $k_{ij} = |\vec{k}_i| + |\vec{k}_j|$ and $k = |\vec{k}_1 + \vec{k}_2|$.

1. This simplicity is explained by IBP [Arkani-Hamed, Benincasa, Postnikov (2017)]

$$\int_0^\infty dt_1 dt_2 \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) e^{ik_{12}t_1} e^{ik_{34}t_2} G(t_1, t_2, \vec{k}) = 0$$

Since (units $1 = i$)

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) e^{ik_{12}t_1} e^{ik_{34}t_2} = (k_{12} + k_{34}) e^{ik_{12}t_1} e^{ik_{34}t_2} \tag{5}$$

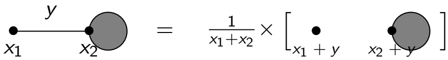
$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) G(t_1, t_2, \vec{k}) = e^{ikt_1} e^{ikt_2} \tag{6}$$

2. We directly get the answer!

$$(k_{12} + k_{34}) I = \int_0^\infty e^{i(k_{12}+k)t_1} dt_1 \times \int_0^\infty e^{i(k_{34}+k)t_2} dt_2 = \frac{1}{(k_{12}+k)(k_{34}+k)} \tag{7}$$

1. Unlike BCFW, these simplifications occur at a diagrammatic level and not at the level of the full correlator.

2. Summary

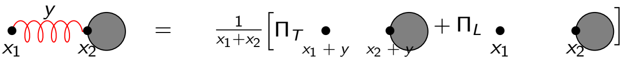


3. These are theories for which there is no apriori BCFW. Possible to write down a one-parameter deformation "BCFW" for each individual diagram (deform $x \to x+z$) [Arkani-Hamed, Baumann, Benincasa, Chen, Joyce, Lee, Pimentel, Postnikov]

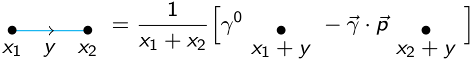
$$\psi(0) = \frac{1}{2\pi i}\oint dz \frac{\psi(z)}{z} = -\sum_j \operatorname*{Res}_{z=z_j} \frac{\psi(z)}{z} + B_\infty \tag{8}$$

Clipping Rules away from $s = 0$?

1. For Gluons we have a nice clipping rule, [Albayrak, CC, Kharel, 2019]



2. We expect something very similar for (massless) fermions too! [CC, Chowdhury, Singh (in progress)]



(9)

1. There exists a BCFW recursion relation for tree-level correlators in YM and Gravity in AdS [Raju, 2011-2012].

2. Simplifies some computations, hides structure, difficult to implement.

3. These also allow one to prove the flat space limit for momentum space quite generally for YM and GR.

4. In 4-dimensions, the answers can be expressed in terms of spinor-helicity variables, [Raju, (2012)]

$$\langle + - + - \rangle = \frac{1}{E} \text{“}\frac{\langle 24\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\text{”} + \mathcal{A}_{\text{pure AdS}}$$

Flat Space limit

1. Translational invariance exists in 3-directions, not in 4-th.
2. Flat space limit $\leftrightarrow$ Restoring translational invariance along 4-th direction

$$\int_0^\infty dte^{iEt} \leftrightarrow \int_{-\infty}^\infty dte^{iEt}$$
$$\implies \frac{1}{E} \leftrightarrow \delta(E)$$

3. Hence the residue at total energy pole gives the flat space amplitude

$$\underset{k_{12}+k_{34}\to 0}{\text{Res}} \frac{1}{(k_{12}+k_{34})(k_{12}+k)(k_{34}+k)} \to \frac{1}{(\vec{k}_1+\vec{k}_2)^2-(|\vec{k}_1|^2+|\vec{k}_2|^2)} = \frac{1}{s}$$

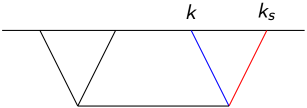
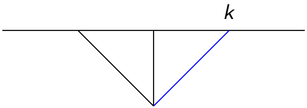
4. Wave function contains the Scattering amplitude!

Some Nice Structure @ Diagrams

1. One nice structure: Double Copy [Armstrong, Goodhew, Lipstein, Mei]

$$\psi_{grav} = \int d\omega \frac{\omega}{\omega^2 + k^2} (KKJ)(KKJ) \left[(YM)^2 + \delta \right]$$

2. Hides soft Limit of Witten diagrams:

$\lim_{k_s \to 0}$  $= \frac{1}{k} \frac{\partial}{\partial k}$ 

Self-Dual Theories

1. For SDYM and SDG, there is a double copy structure at the level of equation of motion. [Lipstein, Nagy (2023)]

2. Implies that there is some double copy at the level of vertices for tree level correlators

$$V_{SDYM} = \{f, g\} \to V_{SDG} = \{\{f, g\}\}$$

where $\{f, g\} = \partial_u f \partial_w g - \cdots$ and $\{\{f, g\}\} = \partial_u^2 f \partial_w g - \cdots$.

3. How much extends to integrated version? Honest answer: Don't know! [CC, Lipstein, Monteiro, Nagy, Singh]

4. Similar soft theorems exist for correlator and one can recover the Weinberg soft theorem in flat limit!

Loops

1. Several calculations exist for loop amplitudes in position space [McFadden, Sachs, Skenderis, Vanhove, $\cdots$]. Some old computations also exist in momentum space for 2-pt at 1-loop [Senatore, Pajer, Lee, $\cdots$].

2. For 4-pt, the bubble diagram at one-loop [Albayrak, CC, Kharel (2020)]

$$= \frac{\lambda_0^2}{8k}\left[\frac{\pi^2}{3} + \frac{4k\log 2}{k_{12}+k_{34}} + \log^2\left(\frac{k_{34}-k}{k+k_{12}}\right) + \log^2\left(\frac{k_{12}-k}{k+k_{34}}\right) - \log^2\left(\frac{k+k_{12}}{k+k_{34}}\right)\right.$$
$$\left. + 2\mathrm{Li}_2\frac{k+k_{34}}{k-k_{12}} + 2\mathrm{Li}_2\frac{k+k_{12}}{k-k_{34}} + \frac{4k}{k_{12}^2-k_{34}^2}\left(k_{34}\log\frac{k+k_{12}}{\Lambda} - k_{12}\log\frac{k+k_{34}}{\Lambda}\right)\right] \tag{10}$$

And 2-pt at 2-loop (cactus, sunset) [CC, Singh (2023)]

$$= \frac{1}{E_T}\left[n_1\Lambda^2 + n_2\Lambda k + n_3 k^2 + n_4 k^2\log\frac{k}{\Lambda} + n_5 k^2\log^2\frac{k}{\Lambda}\right] \tag{11}$$

dS vs Flat @ Loop

1. Scattering amplitudes in flat space are "squares of momenta"

$$\text{Bubble} = \int \frac{d^4 l}{l^2 (l+k)^2}$$

2. Loops in dS are not squares!

$$\text{Generic Loop Integral} = \int \frac{d^3 l}{|\vec{l}+\vec{k}| \times (|\vec{l}| + |\vec{k}| + |\vec{l}+\vec{k}'|) \times \cdots}$$

3. The $|\vec{l}+\cdots|$ appear as they can be thought of as l_0-integrals over flat space

$$\int d^3 l \int_{-\infty}^{\infty} dl_0 \frac{1}{(l_0^2 + \vec{l}^2)\left((l_0+k_0)^2 + (\vec{l}+\vec{k})^2\right)} \qquad (12)$$

This is exactly what happens in the flat space limit.

Correlators from Wave Function

1. Correlation functions can be computed via

$$\langle \phi_1 \cdots \phi_n \rangle = \frac{\int D\phi \, |\Psi[\phi]|^2 \, \phi_1 \cdots \phi_n}{\int D\phi \, |\Psi[\phi]|^2} \tag{13}$$

2. Therefore the cosmological correlator is one more path integral away from the computation of the wave function.

3. Hence it is expected to be more complicated.

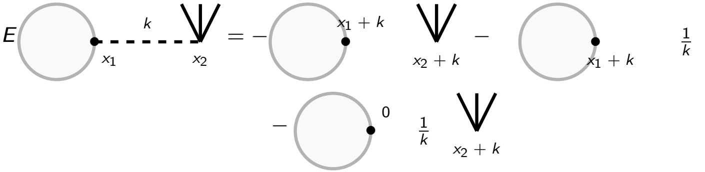
4. BUT for conformally coupled scalar we find that it is simpler than the wave function, due to many cancellations!

5. Not manifest from (13) !

Cosmological Correlator

1. Can be thought of via an Effective action for in-in vertices [Di Pietro, Gorbenko, Komatsu (2021), Heckelbacher, Sachs, Skvortsov, Vanhove (2022), Sleight, Taronna (2021)]

2. Recursion: Along with the solid lines we have dashed lines,



(14)

3. Gives simpler poles for loop integrals of correlators. Upto 2-loops,

$$\langle \phi_1 \cdots \phi_n \rangle = \int \prod_i dp_i R(p_i) \mathrm{Amp}(p_1, \cdots, p_n) \tag{15}$$

Analytic Regularization

1. Loop Integrands can be generated using the recursions/clipping rules. Have to use hard-cutoff to evaluate them.

2. Bad thing: Breaks conformal invariance

3. Analytic regularization [CC, Lipstein, Mei, Sachs, Vanhove (to appear)].

$$\text{Flat Space}: \frac{1}{l^2(l+k)^2} \to \frac{1}{(l^2)^{\kappa_1}\left((l+k)^2\right)^{\kappa_2}} \tag{16}$$

(A)dS

$$(zz')\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2+k^2}\sin(\omega z)\sin(\omega z') \to (zz')^{1-\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{(\omega^2+k^2)^{1+\kappa}}\sin(\omega z)\sin(\omega z') \tag{17}$$

LHS is the $\Theta - \Theta$ form of the Dirichlet Green function. RHS is a Gamma function. Hence recursions are hard to use.

1. Bubble in hard-cutoff

$$\frac{\lambda^2}{32(k_{12}+k_{34})}\left[\ln\left(\frac{(k_{12}+|\vec{k}_{12}|)(k_{34}+|\vec{k}_{12}|)}{4\Lambda^2}\right)+\frac{k_{12}+k_{34}}{k_{12}-k_{34}}\ln\left(\frac{k_{34}+|\vec{k}_{12}|}{k_{12}+|\vec{k}_{12}|}\right)\right] \tag{18}$$

No Li_2 or $\log^2$, hence simpler than Ψ! Above answer does not satisfy CWI.

2. Analytic Regularization

$$\frac{1}{\kappa}\frac{1}{k_{12}+k_{34}} + \frac{\lambda^2}{32(k_{12}+k_{34})}\left[\ln\left(\frac{(k_{12}+|\vec{k}_{12}|)(k_{34}+|\vec{k}_{12}|)}{(k_{12}+k_{34})^2}\right)+\frac{k_{12}+k_{34}}{k_{12}-k_{34}}\ln\left(\frac{k_{34}+|\vec{k}_{12}|}{k_{12}+|\vec{k}_{12}|}\right)\right] \tag{19}$$

Conclusion

1. Find a hint of simplification beyond amplitudes in flat space.
2. Many tools used for studying amplitudes can be generalized to Wave functions/Correlators.
3. Recursive structure for integrands.
4. Manifestly dS invariant regulators for loop integrands.
5. Cosmological correlators are simpler than wave functions at least at conformal coupling.
6. Hint of double copy and soft theorems for simple theories.

Future Directions

1. How much of the story extends to other spins? [Arkani-Hamed, Baumann, Joyce, Pimentel (2018-); Bzowski, Mc Fadden, Skenderis (2013-)]
2. How much can bootstrap do beyond tree level? [Arkani-Hamed, Baumann, Joyce, Pimentel (2018-); Bzowski, Mc Fadden, Skenderis (2013-), Pajer, Wang, · · ·]
3. Recent progress by Arkani-Hamed, et. al for FLRW. How much extends to loops?
4. Witten Diagrams for Supersymmetric theories?
5. Transcendality of Functions at loop?
6. Experimental observables?